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INEQUALITIES FOR WALLIS' PRODUCTS

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Abstract. Some of inequalities for Wallis' products, stronger than inequality of Kazarinoff, are given in this paper.

1. Introduction and results

In 1655 John Wallis proved the formula:

$$\frac{\pi}{2} = \lim_{n\to\infty}\left\{\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \frac{1}{2n+1}\right\} \tag{1}$$

or in equivalent form:

$$\sqrt{\frac{\pi}{2}} = \lim_{n\to\infty}\left\{\frac{(2n)!!}{(2n-1)!!}\cdot\frac{1}{\sqrt{2n+1}}\right\}. \tag{2}$$

Upper products (and them slightly modified), which in limit process determine $\frac{\pi}{2}$ or $\sqrt{\frac{\pi}{2}}$, are known as Wallis' products.

The following inequalities also originate from J. Wallis:

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi n}} \quad (\forall n \in N). \tag{3}$$

In 1956, D. K. Kazarinoff [4] proved that:

$$\frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}} \quad (\forall n \in N). \tag{4}$$

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For Wallis' product $\dfrac{(2n-1)!!}{(2n)!!}$ will be proved some inequalities stronger than the left inequality in (3), and some inequalities stronger than inequality (4).

2. Statements and results

Proposition 1. *By elementary technique can be proved:*

$$\frac{(2n-1)!!}{(2n)!!} > \frac{1}{\sqrt{\pi\left(n+\dfrac{1}{2}\right)}}\sqrt[4]{\frac{2n+2}{2n+1}}, \tag{5}$$

$$\frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi\left(n+\dfrac{1}{4}\right)}}. \tag{6}$$

1. Let

$$I_{2n+1} = \int_0^{\frac{\pi}{2}} \sin^{2n+1} x dx = \int_0^{\frac{\pi}{2}} \sin^n x \sin^{n+1} x dx.$$

If we apply the integral inequality of Cauchy-Bouniakowsky, we get:

$$I_{2n+1} < \sqrt{I_{2n} \cdot I_{2n+2}}.$$

Substituting of expressions for $I_{2n}, I_{2n+2}, I_{2n+1}$ into the last inequality almost immediately yields to (5).

2. Let

$$\begin{aligned} I_{2n} &= \int_0^{\frac{\pi}{2}} \sin^{2n} x dx = \int_0^{\frac{\pi}{2}} \sin^{n-\frac{1}{4}} x \cdot \sin^{n+\frac{1}{4}} x dx < \\ &< \left(\int_0^{\frac{\pi}{2}} \sin^{2n-\frac{1}{2}} x dx \cdot \int_0^{\frac{\pi}{2}} \sin^{2n+\frac{1}{2}} x dx\right)^{\frac{1}{2}} = \sqrt{A_{2n-1} \cdot A_{2n}}, \end{aligned}$$

where is

$$A_n = \int_0^{\frac{\pi}{2}} \sin^{n+\frac{1}{2}} x dx.$$

Using the formula:

$$\int_0^{\frac{\pi}{2}} \sin^k x dx = \frac{1}{2}\mathrm{B}\left(\frac{1}{2}, \frac{k+1}{2}\right) \quad (k > -1),$$

we get:

$$A_{2n-1}\cdot A_{2n}=\frac{1}{2}\mathrm{B}\left(\frac{1}{2},n+\frac{1}{4}\right)\mathrm{B}\left(\frac{1}{2},n+\frac{3}{4}\right)=\frac{\pi}{4n+1}.$$

This straightforwardly implies the inequality (4) $\square$

Remark 1. Inequality (5) is evidently stronger than the left inequality in (3), while the proof of the inequality of Kazarinoff is relatively simple.

Remark 2. It is easy to verify that inequality (5) is equivalent to each of inequalities:

(5') $$I_{2n+1}<\lambda I_{2n},\quad \lambda=\sqrt{\frac{2n+1}{2n+2}};$$

(5") $$I_{2n+1}-I_{2n+2}<\lambda(I_{2n}-I_{2n+1}).$$

Theorem 1.

(7) $$\frac{24n-2}{24n+1}<\sqrt{n\pi}\,\frac{(2n-1)!!}{(2n)!!}<\frac{72n-2}{72n+7}\quad(\forall n\in N).$$

1. First, let us prove the left inequality in (6). We start from inequality (CF. [5]):

(8) $$\left(x+\frac{1}{2}\right)\log\left(1+\frac{1}{x}\right)-1>\frac{1}{12}\left(\frac{1}{x}-\frac{1}{x+1}\right)-\\ -\frac{1}{360}\left(\frac{1}{x^3}-\frac{1}{(x+1)^3}\right)\quad(x>0)$$

If we put $x=k+\frac{1}{2}$, and use the following summation formula (given in the Nielsen's book [1], p. 87,88):

(9) $$\sum_{k=n}^{\infty}\left\{(1+k)\log\left(1+\frac{1}{k+\frac{1}{2}}\right)-1\right\}=\\ =n+\frac{1}{2}-n\log\left(n+\frac{1}{2}\right)+\log\frac{(2n-1)!!}{(2n)!!}-\frac{1}{2}\log 2+\log n!$$

we obtain:

$$
\begin{aligned}
&n+\frac{1}{2}-n\log\left(n+\frac{1}{2}\right)+\log\frac{(2n-1)!!}{(2n)!!}-\frac{1}{2}\log 2+\log n!>\\
&>\frac{1}{12}\cdot\frac{1}{n+\frac{1}{2}}-\frac{1}{360}\cdot\frac{1}{\left(n+\frac{1}{2}\right)^3}=\frac{1}{6}\cdot\frac{1}{2n+1}-\frac{1}{45}\cdot\frac{1}{(2n+1)^3}\quad\Rightarrow\\
&\Rightarrow\quad \log\sqrt{n\pi}\frac{(2n-1)!!}{(2n)!!}>\frac{1}{6}\cdot\frac{1}{2n+1}-\frac{1}{45}\cdot\frac{1}{(2n+1)^3}-\\
&-\left(n+\frac{1}{2}\right)+n\log\left(n+\frac{1}{2}\right)+\frac{1}{2}\log 2+\frac{1}{2}\log n\pi-\log n!
\end{aligned}
\tag{10}
$$

Here we apply inequality of Cesaro-Buchner (CF. [2], p. 154; [3], p. 8-11):

$$
\log n!<\frac{1}{2}\log 2\pi+\left(n+\frac{1}{2}\right)\log n-n+\frac{1}{12n}\quad(\forall n\in N),
\tag{11}
$$

and after introducing:

$$
a_n=\sqrt{n\pi}\frac{(2n-1)!!}{(2n)!!},
$$

we have:

$$
\log a_n>n\log\left(1+\frac{1}{2n}\right)-\frac{1}{2}-\frac{1}{12n(2n+1)}-\frac{1}{45}\cdot\frac{1}{(2n+1)^3}.
\tag{12}
$$

The left inequality in (6) will be proved if we prove that:

$$
\begin{aligned}
&f(n)\equiv n\log\left(1+\frac{1}{2n}\right)-\frac{1}{2}-\frac{1}{12n(2n+1)}-\\
&-\frac{1}{45}\cdot\frac{1}{(2n+1)^3}+\log\frac{24n+1}{24n-2}>0\quad(\forall n\in N).
\end{aligned}
\tag{13}
$$

This inequality can be proved by estimation of signs of $f'(x)$ and $f''(x)$ for $x\geqslant 1$.

2. Now, we are going to prove the right inequality in (6). We start from identity:

$$
\log a_n=\frac{1}{2}\log n\pi+\log(2n)!-2n\log 2-2\log n!
$$

and apply Cesaro-Buchner's inequalities:

$$
\log n!<\frac{1}{2}\log 2\pi+\left(n+\frac{1}{2}\right)\log n-n+\frac{1}{12n}\quad(\forall n\in N),
\tag{14}
$$

$$
\log n!>\frac{1}{2}\log 2\pi+\left(n+\frac{1}{2}\right)\log n-n+\frac{1}{12n+\frac{1}{4}}\quad(\forall n\in N)
\tag{15}
$$

in appropriate places, guided by the rule of maximizing the right side of identity. Hence we obtain:

$$\log a_n < \frac{1}{24n} - \frac{2}{12n + \frac{1}{4}}.$$

The rest to be proved is inequality:

$$g(n) \equiv \log \frac{72n-2}{72n+7} - \frac{1}{24n} + \frac{2}{12n + \frac{1}{4}} > 0 \quad (\forall n \in N).$$

Estimation of sign of $g'(x)$ for $x \geqslant 1$ lead us to the desired result. □

Remark 3. Left and right inequality in (6) are respectively stronger than inequalities (5) and (4).

Remark 4. Inequalities (3) and (4) taken together imply:

(16) $$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < \frac{(2n-1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}} \quad (\forall n \in N).$$

Corollary 1.

(17) $$\frac{(2n-1)!!}{(2n)!!} > \frac{1}{\sqrt{\pi\left(n+\frac{1}{3}\right)}} \quad (\forall n \in N).$$

Let us rewrite the left inequality in (6):

(6') $$\frac{(2n-1)!!}{(2n)!!} > \frac{1}{\sqrt{\pi n}} \cdot \frac{24n-2}{24n+1} \quad (\forall n \in N).$$

The following inequality:

(18) $$\frac{1}{\sqrt{n\pi}} \cdot \frac{24n-2}{24n+1} > \frac{1}{\sqrt{\pi(n+\frac{1}{3})}}$$

holds true for every n □

Remark 5. We have inequality stronger than (17), namely:

(19) $$\frac{(2n-1)!!}{(2n)!!} > \frac{1}{\sqrt{\pi\left(n+\frac{7}{24}\right)}} \quad (\forall n \in N).$$

Corollary 2.

$$\sqrt{\frac{2}{\pi}}\cdot\frac{8n+\frac{1}{2}}{8n}<\sqrt{2n+1}\frac{(2n-1)!!}{(2n)!!}<\sqrt{\frac{2}{\pi}}\cdot\frac{8n+1}{8n}\quad(\forall n\in N). \tag{20}$$

$$\sqrt{2n+1}\cdot\frac{(2n-1)!!}{(2n)!!}\overset{\substack{\text{left inequality}\\ \text{in (6)}}}{>}\sqrt{2n+1}\cdot\frac{1}{\sqrt{n\pi}}\cdot\frac{24n-2}{24n+1}>\sqrt{\frac{2}{\pi}}\cdot\frac{8n+\frac{1}{2}}{8n}.$$

The last inequality can be verified by direct computation.

$$\sqrt{2n+1}\cdot\frac{(2n-1)!!}{(2n)!!}\overset{\substack{\text{right inequality}\\ \text{in (6)}}}{<}\sqrt{2n+1}\cdot\frac{1}{\sqrt{n\pi}}\cdot\frac{72n-2}{72n+7}<\sqrt{\frac{2}{\pi}}\cdot\frac{8n+1}{8n}.$$

The last inequality can be easily proved. □

Finally, we give asymptotic formula for Wallis' product:

$$\begin{aligned}&\sqrt{2n+1}\frac{(2n-1)!!}{(2n)!!}=\\ &=\sqrt{\frac{2}{\pi}}\left(1+\frac{1}{8n}-\frac{7}{128n^2}+\frac{19}{1024n^3}-\frac{303}{3\cdot 8^5n^4}+\mathrm{o}\left(\frac{1}{n^4}\right)\right)\quad(n\to\infty).\end{aligned} \tag{21}$$

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3. References

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